UDC 62 —50

QUASILINEAR ENCOUNTER PROBLEM WITH PARTICIPATION OF SEVERAL PERSONS

PMM Vol. 43, No. 3, 1979, pp. 451—455

A. A. CHIKRII
(Kiev)

(Received February 24, 1978)

A game problem is analyzed of the encounter of a strictly linear system with a terminal set with a strong structure, being the union of a finite number of convex sets. In particular, this problem includes the problem of the pursuit of an escaping object by several controlled objects. Two types of sufficient solvability conditions are obtained for the encounter problem. The paper is closely related to the investigations in [1 – 4].

The motion of an object in a finite-dimensional Euclidean space E^n can be described by the equation

$$z^{\cdot} = Az + \varphi(u, v), \quad u \in U, \quad v \in V \tag{1}$$

where A is a square $n \times n$ -matrix, $\varphi(u, v)$ is a function continuous in all its arguments, and U and V are compacta in E^n . The terminal set M is the union of sets $M_1^*, \ldots, M_\nu^*$, where each of the M_i^* has the form $M_i^* = M_i^\circ + M_i$; M_i° are linear subspaces of E^n, and M_i are closed convex sets belonging to the orthogonal complements of M_i° in E^n. The problem is to find conditions on the parameters of game (1) that ensure the encounter of system (1) from a specified initial position with terminal set M (the end of game (1)) in finite time. We say that game (1) with $z(0) = z_0$ can be completed in finite time if a number $T(z_0)$ exists and from the current state $z(t)$ and $v(t)$ we can construct a measurable function $u(t)$, $u(t) \in U$, $0 \leqslant t \leqslant T(z_0)$, where $v(t)$ is an arbitrary piecewise-continuous function with values from V , such that the inclusion $z(t_1) \in M$, where $t_1 \leqslant T(z_0)$, holds for the corresponding trajectories [2].

By L_i we denote the orthogonal complement of M_i° in E^n and by π_i we denote the operator of orthogonal projection from E^n onto L_i. Then the inclusion $z(t) \in M$ is equivalent to the existence of a number i such that $\pi_i z(t) \in M_i$. The support function of set M_i^* has the form

$$W_{M_i^*}(\psi) = \begin{cases} \sup\limits_{z \in M_i} (z, \psi), & \psi \in L_i \\ \infty, & \psi \overline{\in} L_i \end{cases}$$

We assume that the sets $K(M_i) = \{\psi : \psi \in L_i, W_{M_i}(\psi) < \infty\}$ are nonempty and closed and that the functions $W_{M_i}(\psi)$ are continuous on $K(M_i)$, $i = 1, \ldots, \nu$. We assume as well that set $\varphi(U, v)$ is convex for any $v \in V$.

Let $N_i(q)$, $i = 1, \ldots, \nu$, be continuous bounded convex-valued mappings with values in 2^{E^n}, defined on some compactum Q, and let X_i be closed convex sets from L_i , where the functions $W_{X_i}(\psi)$ are continuous on $K(X_i)$. We denote

$$\lambda_i(q) = \min_{\|\psi\|=1, \psi \in L_i} [W_{N_i(q)}(\psi) + W_{X_i}(-\psi)]$$

We see that the minimum over ψ is achieved on elements of set $-K(X_i)$.

Lemma 1. For an element $q^* \in Q$ to exists such that

$$\bigcap_{i=1}^{\nu} (N_i(q^*) \cap X_i) = \phi$$

it is necessary and sufficient that

$$\min_{q \in Q} \max_{i=1,\ldots,\nu} \lambda_i(q) < 0$$

The proof follows from Theorem 1.1. in [2].

We form the sets

$$\Psi = \{\psi : \psi = (\psi_1, \ldots, \psi_\nu), \| \psi_i \| = 1, \psi_i \in L_i\}$$

$$H = \left\{\alpha : \alpha = (\alpha_1, \ldots, \alpha_\nu), \alpha_i \geqslant 0, \sum_{i=1}^{\nu} \alpha_i = 1, \alpha_i \in E^1\right\}$$

and we consider attainability set of system (1) from an initial position z in the time t for a fixed control $v(\cdot)$

$$D(t, z, v(\cdot)) = \Phi(t) z + \int_0^t \Phi(t-\tau) \varphi(U, v(\tau)) \, d\tau$$

$$\Phi^{\cdot}(t) = A\Phi(t), \quad \Phi(0) = I$$

See [4] with regard to the integration of convex sets. The set $D(t, z, v(\cdot))$ is closed and convex. Having projected set $D(t, z, v(\cdot))$ onto a subspace L_i and computed the support function of the resultant set, we introduce the following functions into consideration:

$$\lambda_i(t, z, v(\cdot)) = \min_{\|\psi_i\|=1} [W_{D(t, z, v(\cdot))}(\psi_i) + W_{M_i}(-\psi_i)], \quad i = 1, \ldots, \nu$$

$$\lambda(t, z) = \min_{v(\cdot)} \max_{i=1,\ldots,\nu} \lambda_i(t, z, v(\cdot))$$

We see that $z \in M$ if and only if $\lambda(0, z) < 0$. For $\psi \in \Psi$, $\alpha \in H$ we set

$$W(t, z, \alpha, \psi) = \sum_{i=1}^{\nu} \alpha_i (\Phi^*(t) \psi_i, z) + \int_0^t \min_{v \in V} \max_{u \in U} \sum_{i=1}^{\nu} (\alpha_i \Phi^*(\tau) \psi_i, \varphi(u, v)) \, d\tau$$

$$\mu(t, z, \alpha, \psi) = W(t, z, \alpha, \psi) + \sum_{i=1}^{\nu} \alpha_i W_{M_i}(-\psi_i)$$

$$\lambda^*(t, z) = \min_{\psi \in \Psi} \max_{\alpha \in H} \mu(t, z, \alpha, \psi), \quad \lambda_*(t, z) = \max_{\alpha \in H} \min_{\psi \in \Psi} \mu(t, z, \alpha, \psi)$$

We denote

$$\Psi' = \{\psi : \psi = (\psi_1, \ldots, \psi_\nu),\ \|\psi_i\| \leqslant 1,\ \psi_i \in L_i\}$$

Let

$$\min_{\psi\in\Psi}\max_{\alpha\in H} \mu(t, z, \alpha, \psi) = \min_{\psi\in\Psi'}\max_{\alpha\in H} \mu(t, z, \alpha, \psi) \tag{2}$$

L e m m a 2. The inequality

$$\lambda(t, z) \geqslant \lambda_*(t, z), \quad t > 0,\ z \in E^n$$

holds. If equality (2) is fulfilled, then

$$\lambda(t, z) \geqslant \lambda^*(t, z), \quad t \geqslant 0, \quad z \in E^n \tag{3}$$

P r o o f. If $\lambda_1, \ldots, \lambda_\nu$ are certain numbers, then the equality

$$\max_{\alpha\in H} \sum_{j=1}^{\nu} \alpha_i \lambda_i = \max_{i=1,\ldots,\nu} \lambda_i \tag{4}$$

is valid. Then, using the minimax theorem [5], the inequality connecting minimax with maximin, and using (2), we obtain

$$\lambda(t, z) = \min_{v(\cdot)}\max_{\alpha\in H} \sum_{i=1}^{\nu} \alpha_i \lambda_i(t, z, v(\cdot)) \geqslant \min_{\psi\in\Psi'}\min_{v(\cdot)}\max_{\alpha\in H} \sum_{i=1}^{\nu} (\alpha_i W_{D(t, z, v(\cdot))}(\psi_i) +$$

$$\alpha_i W_{M_i}(-\psi_i)) \geqslant \min_{\psi\in\Psi}\max_{\alpha\in H} \mu(t, z, \alpha, \psi) = \lambda^*(t, z)$$

We set

$$W(t, z, \psi_i) = (\Phi^*(t)\psi_i, z) + \int_0^t \min_{v\in V}\max_{u\in U} (\Phi^*(\tau)\psi_i, \varphi(u, v))\, d\tau$$

$$\lambda_i(t, z) = \min_{\|\psi_i\|=1} [W(t, z, \psi_i) + W_{M_i}(-\psi_i)], \quad \psi_i \in L_i, \quad i = 1, \ldots, \nu$$

A similar function was introduced in [2]. Let

$$\max_{u\in U} \sum_{i=1}^{\nu} (\alpha_i \Phi^*(t)\psi_i, \varphi(u, v)) = \tag{5}$$

$$\sum_{i=1}^{\nu} \max_{u\in U} (\alpha_i \Phi^*(t)\psi_i, \varphi(u, v)), \quad t \geqslant 0, \quad \alpha \in H, \quad \psi \in \Psi, \quad v \in V$$

L e m m a 3. Let equality (5) be fulfilled. Then

$$\lambda_*(t, z) \geqslant \max_{i=1,\ldots,\nu} \lambda_i(t, z), \quad t \geqslant 0, \quad z \in E^n \tag{6}$$

P r o o f. Since

$$\min_{v\in V} \sum_{i=1}^{\nu} \max_{u\in U} (\alpha_i \Phi^*(\tau)\psi_i, \varphi(u, v)) \geqslant \sum_{i=1}^{\nu} \min_{v\in V}\max_{u\in U} (\alpha_i \Phi^*(\tau)\psi_i, \varphi(u, v))$$

then, by virtue of equality (4), from (5) we have

$$\mu(t, z, \alpha, \psi) \geqslant \sum_{i=1}^{\nu} \alpha_i [W(t, z, \psi_i) + W_{M_i}(-\psi_i)]$$

$$\lambda_*(t, z) \geqslant \max_{\alpha \in H} \min_{\psi \in \Psi} \sum_{i=1}^{\nu} \alpha_i [W(t, z, \psi_i) + W_{M_i}(-\psi_i)] = \max_{i=1,\ldots,\nu} \lambda_i(t, z)$$

We define functions $T^*(z)$ and $T_*(z)$ as follows: function $T^*(z)$ $(\check{T}_*(z))$ equals the greatest lower bound of the roots of the equation $\lambda^*(t, z) = 0$ $(\lambda_*(t, z) = 0)$, $t \geqslant 0$. If the equation has no roots, we set $T^*(z)$ $(T_*(z))$ equal to $+\infty$. We see that $T^*(z) \leqslant T_*(z)$. The next Lemma reveals the true sense of function $T^*(z)$.

Lemma 4. Let $T^*(z^\circ) < +\infty$ and let equality (2) be fulfilled. Then for any measurable function $v(t) \in V$, $0 \leqslant t \leqslant T^*(z^\circ)$, a measurable function $u(t) \in U$ exists such that $z(T^*(z^\circ)) \in M$.

The proof follows from Lemmas 1 and 2.

In what follows we shall derive two types of sufficient conditions ensuring the end of game (1) by the time $T^*(z)$. Similar conditions can be obtained for the fime $T_*(z)$. We consider the many-valued mappings

$$H(t, z, \psi) = \{\alpha\colon \mu(t, z, \alpha, \psi) = \max_{\alpha \in H} \mu(t, z, \alpha, \psi)\}$$

$$\Psi(t, z) = \{\psi\colon \max_{\alpha \in H} \mu(t, z, \alpha, \psi) = \lambda^*(t, z)\}$$

and we introduce the function

$$\beta(t, \psi, \alpha, u, v) = \sum_{i=1}^{\nu} \alpha_i (\Phi^*(t)\psi_i, \varphi(u, v)) - \min_{v \in V} \max_{u \in U} \sum_{i=1}^{\nu} (\alpha_i \Phi^*(t)\psi_i, \varphi(u, v))$$

In what follows we assume that mapping $H(t, z, \psi)$ is continuous in ψ on set $\Psi(t, z)$ for fixed (t, z). We assume the fulfilment of the second inequality from Lemma 2.

Theorem 1. Let the conditions

a) $$\min_{v \in V} \max_{u \in U} \sum_{i=1}^{\nu} (\alpha_i \Phi^*(T^*(z))\psi_i, \varphi(u, v)) > 0$$

$$\forall \psi \in \Psi(T^*(z), z), \ \alpha \in H(T^*(z), z, \psi);$$

b) for each v° there exists u° such that

$$\beta(T^*(z), \psi, \alpha, u^\circ, v^\circ), \geqslant 0$$

$$\forall \psi \in \Psi(T^*(z), z), \ \alpha \in H(T^*(z), z, \psi)$$

be fulfilled at each point z for which $0 < T^*(z) < \infty$. Then game (1), starting from point z_0, $T^*(z_0) < \infty$, can be completed in a time no greater than $T^*(z_0) + \varepsilon$, where ε is an arbitrary positive number.

T h e o r e m 2. Let the conditions

a) set $\Psi(t, z)$ consists of a single vector $\psi(t, z)$ for all t and z from some neighborhood of $\{T^*(z^1), z^1\}$;

b) set $H(t, z, \psi(t, z))$ consists of a single element $\alpha(t, z)$ for all t and z from the neighborhood mentioned;

c) the maximum of the expression

$$\sum_{i=1}^{\nu} \alpha_i(t, z)(\Phi^*(t)\psi_i(t, z), \varphi(u, v))$$

over u is achieved on a single vector $u(t, z, v)$ for all t and z from the neighborhood mentioned and $v \in V$, be fulfilled at each point z^1 for which $0 < T^*(z^1) < \infty$. Then game (1) can be completed in a time not exceeding $T^*(z_0)$, where z_0 is the initial position of game (1), under any piecewise-continuous control $v(t)$ applied by the opponent.

N o t e. Theorem 2 is true even without assumption b), but then condition c) must be fulfilled with some $\alpha \in H(T^*(z^1), z^1, \psi(T^*(z^1), z^1))$.

Theorems 1 and 2 are proved similarly as in [6] by the methods suggested in [2].

An interesting case is when relation (6) is fulfilled as a strict inequality. The following example of a pursuit problem illustrates this situation.

E x a m p l e. The motion of pursuers and escaping are described by the equations

$$x_i^{\cdot} = u_i, \ \|u_i\| \leqslant a, \ i = 1,2, \quad x_i \in E^n, \ n \geqslant 2$$
$$y^{\cdot} = v, \quad \|v\| \leqslant 1, \ a > 1, \quad y \in E^n$$

The pursuit ends at the instant that one of the equalities $\|x_i(t) - y(t)\| \leqslant \varepsilon$, $i = 1,2$ is fulfilled. Assuming that the players' initial positions satisfy the conditions

$$\|x_1^\circ - y^\circ\| = \|x_2^\circ - y^\circ\|, \quad \varepsilon > 1/2\|x_1^\circ - x_2^\circ\|, \quad \frac{x_1^\circ - y^\circ}{\|x_1^\circ - y^\circ\|} \neq \frac{x_2^\circ - y^\circ}{\|x_2^\circ - y^\circ\|}$$

and carrying out the corresponding calculations, we obtain the inequality

$$\lambda_*(t, z^\circ) > \lambda_1(t, z^\circ) = \lambda_2(t, z^\circ), \quad z^\circ = (x_1^\circ, x_2^\circ, y^\circ)$$

and, consequently

$$T^*(z^\circ) < \frac{\|x_i^\circ - y^\circ\| - \varepsilon}{a - 1}, \quad i = 1, 2$$

In this example conditions b) and c) of Theorem 2 are fulfilled, while condition a) is fulfilled for the initial positions and all current positions arising in this game if control u is chosen from condition c) of Theorem 2.

REFERENCES

1. K r a s o v s k i i, N. N. and S u b b o t i n, A. I., Position Differential Games. Moscow, "Nauka", 1974.

2. P s h e n i c h n y i, B. N., Linear differential games. Avtom. Telemekh., № 1, 1968. (see also Topics in Differential Games, Blaquière, New York, London, Amsterdam, North Holland Publ. Co., 1973).

3. T a r l i n s k i i, S. L., On a linear differential game of the encounter of several controlled objects. Dokl. Akad. Nauk SSSR, Vol. 230, No. 3, 1976.

4. P o n t r i a g i n, L. S., On linear differential games. 2. Dokl. Akad. Nauk SSSR, Vol. 175, No. 4, 1967.

5. D e m ' i a n o v, V. F. and M a l o z e m o v, V. N., Introduction to Minimax. Moscow, "Nauka", 1972.

6. C h i k r i i, A. A. and R a p p o p o r t, I. S., Linear problem of pursuit by several controlled objects. Kibernetika, No. 3, 1978.

Translated by N. H. C.